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*The Complementary Theorem.**

BY J. C. FIELDS.

Let

$$f(z, u) = u^n + f_{n-1}u^{n-1} + \dots + f_0 = 0 \quad (1)$$

be an algebraic equation in which the coefficients f_s are rational functions of z , integral or fractional. The equation may be reducible or irreducible, but is supposed to be rid of repeated factors. In the neighborhood of a value $z = a$ the functions u defined by the equation will be represented by n series in powers of $z - a$ with exponents which may be integral or fractional, positive or negative, the number of the negative exponents, however, being in any case finite. These n series, or branches of the equation, as we shall call them, group themselves into a number r of cycles of orders $\nu_1, \nu_2, \dots, \nu_r$ respectively. A cycle of order ν consists of ν branches developable in integral powers of $(z - a)^{\frac{1}{\nu}}$ and successively passing into one another on making the circuit of the point $z = a$ in the z -plane. For the application of the preceding remarks to the value $z = \infty$ it is only necessary to replace the element $z - a$ by $\frac{1}{z}$. The order of a cycle is in general 1, and the cycle consists of a single branch whose development involves only integral exponents. A cycle of order 1 we might call a simple cycle. The number of values of the variable z to which correspond other than simple cycles is finite.

If now in a rational function $H(z, u)$ we substitute for u the series which represents a certain branch corresponding to a value $z = a$, the result will be representable as a series in powers of $z - a$ (or $\frac{1}{z}$), and the lowest exponent in this series we will call the order of coincidence of the function with the branch in question. The order of coincidence of a rational function of (z, u) with a

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branch of the equation $f(z, u) = 0$ corresponding to a given value of the variable z may be positive or negative; in general, however, it will be 0. Evidently the orders of coincidence of a rational function of (z, u) with different branches of the same cycle will all be equal. The number of cycles with whose branches the orders of coincidence of a given rational function of (z, u) are different from 0 is finite, and the aggregate sum of the orders of coincidence of the function with all the branches of all these cycles is equal to 0. We might then say, for conciseness, that the aggregate sum of all the orders of coincidence of any given rational function of (z, u) with all the branches of all the cycles corresponding to all values of the variable z , the value $z = \infty$ included, is equal to 0. The orders of coincidence of a rational function of (z, u) with the branches of the r cycles corresponding to a given value $z = a$ will evidently be integral multiples of the corresponding fractions $\frac{1}{v_1}, \frac{1}{v_2}, \dots, \frac{1}{v_r}$ respectively.

We shall call a system of sets of numbers $\tau_1^{(k)}, \tau_2^{(k)}, \dots, \tau_{r_k}^{(k)}$ associated with the different values of the variable $z = a_k$, and multiples of the corresponding fractions $\frac{1}{v_1^{(k)}}, \frac{1}{v_2^{(k)}}, \dots, \frac{1}{v_{r_k}^{(k)}}$, a basis of coincidences for the building of a rational function of (z, u) . We conceive a set of numbers to be associated with each value of the variable, the numbers however being all 0 save in the sets associated with a finite number of values of z . When therefore we specify the sets associated with a finite number of values of the variable only, it is to be understood that the sets associated with the remaining values of the variable are all made up of zeros. By the most general rational function built on a given basis of coincidences we mean the most general rational function whose orders of coincidence with the branches of the several cycles corresponding to the various values of z do not fall short of the corresponding numbers mentioned in the basis. It might happen that the most general function built on a proposed basis is identically 0, in which case we call the basis an impossible one. To briefly designate the basis of coincidences here in question we employ the notation (τ) .

A basis $(\bar{\tau})$ we say is complementary to the basis (τ) with regard to the level furnished by a rational function $K(z, u)$ when its numbers $\bar{\tau}_1^{(k)}, \bar{\tau}_2^{(k)}, \dots, \bar{\tau}_{r_k}^{(k)}$ are connected with the numbers $\tau_1^{(k)}, \tau_2^{(k)}, \dots, \tau_{r_k}^{(k)}$ by the equations

$$\tau_s^{(k)} + \bar{\tau}_s^{(k)} = m_s^{(k)} - 1 + \frac{1}{v_s^{(k)}} \quad (s = 1, 2, \dots, r_k) \quad (2)$$

for all finite values of the variable $z = a_k$, and for the value $z = \infty$ by the equations

$$\tau_s^{(\infty)} + \bar{\tau}_s^{(\infty)} = m_s^{(\infty)} + 1 + \frac{1}{\nu_s^{(\infty)}} \quad (s = 1, 2, \dots, r_\infty), \quad (3)$$

where $m_1^{(k)}$, $m_2^{(k)}$, $\dots$, $m_{r_k}^{(k)}$ represent the actual orders of coincidence of the rational function $K(z, u)$ corresponding to the value $z = a_k$. The basis (τ) is then evidently also complementary to the basis $(\bar{\tau})$ with regard to the level furnished by the function $K(z, u)$.

On employing the symbols N_τ and $N_{\bar{\tau}}$ to designate the numbers of arbitrary constants involved in the expressions for the most general rational functions built on the complementary bases (τ) and $(\bar{\tau})$, respectively, the statement of the complementary theorem is contained in the equation

$$N_\tau + \frac{1}{2} \sum_k \sum_{s=1}^{r_k} \tau_s^{(k)} \nu_s^{(k)} = N_{\bar{\tau}} + \frac{1}{2} \sum_k \sum_{s=1}^{r_k} \bar{\tau}_s^{(k)} \nu_s^{(k)}, \quad (4)$$

where the summations with regard to k are supposed to extend to all values of the variable z , the value $z = \infty$ included.

In the definition of complementary bases given above, it would at first sight appear as if the value $z = \infty$ had been treated differently from the finite values of the variable. The difference in treatment is however only apparent, and the complementary bases (τ) and $(\bar{\tau})$ could be symmetrically defined with regard to all values of the variable z , the value $z = \infty$ included, by the system of equations

$$\tau_s^{(k)} + \bar{\tau}_s^{(k)} = \bar{m}_s^{(k)} + \varepsilon^{(k)} - 1 + \frac{1}{\nu_s^{(k)}} \quad (s = 1, 2, \dots, r_k), \quad (5)$$

where the numbers $\varepsilon^{(k)}$ are all zero, save a finite number, which are integers positive or negative, subject only to the condition that $\sum_k \varepsilon^{(k)} = 2$; and where the numbers $\bar{m}_1^{(k)}$, $\bar{m}_2^{(k)}$, $\dots$, $\bar{m}_{r_k}^{(k)}$ represent the actual orders of coincidence of a rational function $\bar{K}(z, u)$ which is connected with the function $K(z, u)$ by the relation

$$K(z, u) = \prod_k (z - a_k)^{\varepsilon^{(k)}} \cdot \bar{K}(z, u),$$

where the product is extended to all finite values of the variable $z = a_k$.

By a transformation $v = gu$, where g is an integral rational function of z , we can make the equation $f(z, u) = 0$ go over into an equation

$$F(z, v) = v^n + F_{n-1} v^{n-1} + \dots + F_0 = 0, \quad (6)$$

in which the coefficients F_s are integral rational functions of z . Rational

functions of (z, u) are rational functions of (z, v) , and conversely; orders of coincidence are unchanged by the transformation, the cycles remain the same, and complementary bases are still complementary bases. When then we have proved the complementary theorem for the equation $F(z, v) = 0$, we have evidently at the same time proved it also for the equation $f(z, u) = 0$. In my book* on the algebraic functions I have given a proof of the complementary theorem which involves a number of more or less complicated formulae† and certain details which can be dispensed with on introducing modifications in the reasoning there employed. The object of the present paper is to give the modified proof here referred to. Before proceeding with the proof, however, it will be necessary, to recall certain results obtained in the earlier half of my book, and to make reference to the notation which I have employed.

Starting out from the integral algebraic equation $F(z, v) = 0$, we shall consider rational functions of (z, v) . Any such function can be written in the form

$$H(z, v) = h_{n-1} v^{n-1} + \dots + h_0, \quad (7)$$

where the coefficients h_s are rational functions of z . This is called the reduced form of the function, and we shall suppose in this paper that a rational function is always expressed in its reduced form excepting where the context implies the contrary.

The orders of coincidence of the rational function $F'_v(z, v)$ with the branches of the r_k cycles corresponding to a value $z = a_k$ we indicate by the symbols $\mu_1^{(k)}, \mu_2^{(k)}, \dots, \mu_{r_k}^{(k)}$ respectively, and by the symbols $\mu_1^{(\infty)}, \mu_2^{(\infty)}, \dots, \mu_{r_\infty}^{(\infty)}$ respectively its orders of coincidence with the branches of the r_∞ cycles corresponding to the value $z = \infty$. If the orders of coincidence of a rational function of (z, v) with the branches of the r_k cycles corresponding to a given value of the variable z do not fall short of the numbers

$$\mu_1^{(k)} - 1 + \frac{1}{v_1^{(k)}}, \dots, \mu_{r_k}^{(k)} - 1 + \frac{1}{v_{r_k}^{(k)}} \quad (8)$$

respectively, we say that the function is adjoint relatively to the equation $F(z, v) = 0$ for the value of the variable in question, and the coincidences are said to constitute a set of adjoint orders of coincidence. This definition extends

* Theory of the Algebraic Functions of a Complex Variable. Mayer & Müller, Berlin, 1906. In the present paper this book will be cited under the letters *A. F.*

† *A. F.*, Chapter XI. The proof here given displaces the greater part of Chapter XI and modifies the beginning of the chapter following.

not only to all finite values of the variable, but also to the value $z = \infty$. If the product of two functions is adjoint for a given value of the variable, we say that the functions are complementary adjoint for the value of the variable in question; and if the orders of coincidence of the product with the branches of the r_k cycles are equal to or greater than the corresponding numbers

$$i + \mu_1^{(k)} - 1 + \frac{1}{\nu_1^{(k)}}, \dots, i + \mu_{r_k}^{(k)} - 1 + \frac{1}{\nu_{r_k}^{(k)}}, \quad (9)$$

where i is any positive integer, we say that the functions are complementary adjoint to the order i .

In considering a rational function of (z, v) with reference to the cycles corresponding to the value $z = \infty$, it is at times convenient to transform to the variables $\xi = z^{-1}$, $\eta = z^{-m}v$, where z^m is the highest power of z which appears in the equation $F(z, v) = 0$. This equation then goes over into the integral algebraic equation

$$G(\xi, \eta) = 0, \quad (10)$$

in which η is the dependent variable. The rational function

$$G'_\eta(\xi, \eta) = \xi^{m(n-1)} F'_v(z, v)$$

evidently has, as orders of coincidence with the branches of the cycles corresponding to the value $\xi = 0$, the numbers

$$m(n-1) + \mu_1^{(\infty)}, \dots, m(n-1) + \mu_{r_\infty}^{(\infty)}; \quad (11)$$

and the orders of coincidence defining adjointness relatively to the equation $G(\xi, \eta) = 0$ for the cycles corresponding to the value $\xi = 0$, are

$$m(n-1) + \mu_1^{(\infty)} - 1 + \frac{1}{\nu_1^{(\infty)}}, \dots, m(n-1) + \mu_{r_\infty}^{(\infty)} - 1 + \frac{1}{\nu_{r_\infty}^{(\infty)}} \quad (12)$$

respectively. When the product of two rational functions has orders of coincidence for the value $\xi = 0$ which do not fall short of these numbers, they are complementary adjoint relatively to the equation $G(\xi, \eta) = 0$ for the value of ξ here in question; and when the orders of coincidence of the product do not fall short of the numbers

$$m(n-1) + i + \mu_1^{(\infty)} - 1 + \frac{1}{\nu_1^{(\infty)}}, \dots, m(n-1) + i + \mu_{r_\infty}^{(\infty)} - 1 + \frac{1}{\nu_{r_\infty}^{(\infty)}} \quad (13)$$

respectively, the functions are complementary adjoint to the order i relatively to the equation (10) for the value $\xi = 0$.

A rational function of (z, v) can be written in the form

$$H(z, v) = P(z, v) + \sum_k \frac{\phi^{(i_k)}(z, v)}{(z - a_k)^{i_k}}, \quad (14)$$

where the expressions $P(z, v)$ and $\phi^{(i_k)}(z, v)$ represent polynomials in (z, v) , reduced with regard to v , and where a polynomial $\phi^{(i_k)}(z, v)$ is of degree less than i_k in z . Here the summation is supposed to extend to all those finite values $z = a_k$ for which the function $H(z, v)$ may become infinite, as also to all those values of the variable to which correspond multiple points on the curve $F(z, v) = 0$. Transformed to terms of (ξ, η) our rational function can also be represented in the form

$$H(z, v) = -\frac{\phi^{(i_\infty + j)}(\xi, \eta)}{\xi^{i_\infty}} + \xi^j((\xi, \eta)), \quad (15)$$

where $\phi^{(i_\infty + j)}(\xi, \eta)$ is a polynomial in (ξ, η) reduced in η , and of a degree in ξ which is less than $i_\infty + j$. Here j is an integer which is at our disposal, and the expression $((\xi, \eta))$ signifies a reduced polynomial in η with coefficients which, expanded in powers of ξ , involve no negative exponents.

If now we characterize the function $H(z, v)$ by saying that it is to have orders of coincidence with the branches of the r_∞ cycles corresponding to the value $z = \infty$, which do not fall short of the numbers $\tau_1^{(\infty)}, \tau_2^{(\infty)}, \dots, \tau_{r_\infty}^{(\infty)}$ respectively, while it is to become infinite to orders not exceeding $\sigma_1^{(k)}, \sigma_2^{(k)}, \dots, \sigma_{r_k}^{(k)}$ respectively for the branches of the r_k cycles corresponding to a value $z = a_k$, we shall have for any exponent i_k in formula (14) the greatest of the integers*

$$[\mu_1^{(k)} + \sigma_1^{(k)}], \dots, [\mu_{r_k}^{(k)} + \sigma_{r_k}^{(k)}], \quad (16)$$

and for the exponent i_∞ in formula (15) the greatest of the integers†

$$m(n - 1) + [\mu_1^{(\infty)} + \sigma_1^{(\infty)}], \dots, m(n - 1) + [\mu_{r_\infty}^{(\infty)} + \sigma_{r_\infty}^{(\infty)}], \quad (17)$$

where corresponding numbers in the sets $\sigma_1^{(\infty)}, \dots, \sigma_{r_\infty}^{(\infty)}$ and $\tau_1^{(\infty)}, \dots, \tau_{r_\infty}^{(\infty)}$ are equal and of opposite signs excepting where a number τ is positive, in which case the corresponding number σ has the value 0.

It is to be understood that the integer j in formula (15) is not negative and furthermore we shall find it convenient to assume that it is not less than the greatest of the numbers $\tau_1^{(\infty)}, \tau_2^{(\infty)}, \dots, \tau_{r_\infty}^{(\infty)}$, so that the coefficients in the expression $\xi^j((\xi, \eta))$ will not be conditioned by the orders of coincidence of $H(z, v)$ for the value $z = \infty$. A numerator $\phi^{(i_k)}(z, v)$ appearing in the summation in

* A. F., X.

† A. F., XI, 6.

formula (14) will have orders of coincidence with the branches of the r_k corresponding cycles which do not fall short of the numbers

$$i_k - \sigma_1^{(k)}, i_k - \sigma_2^{(k)}, \dots, i_k - \sigma_{r_k}^{(k)} \quad (18)$$

respectively, and will involve ρ_k arbitrary constants, where*

$$\rho_k = \sum_{s=1}^{r_k} \sigma_s^{(k)} \nu_s^{(k)} + \frac{1}{2} \sum_{s=1}^{r_k} \left(\mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}} \right) \nu_s^{(k)}. \quad (19)$$

The numerator $\phi^{(i_\infty+j)}(\xi, \eta)$ appearing on the right-hand side of formula (15) will have orders of coincidence with the branches of the r_∞ cycles corresponding to the value $z = \infty$, which do not fall short of the numbers

$$i_\infty + \tau_1^{(\infty)}, i_\infty + \tau_2^{(\infty)}, \dots, i_\infty + \tau_{r_\infty}^{(\infty)} \quad (20)$$

respectively, and will involve ρ_∞ arbitrary constants, where†

$$\rho_\infty = \sum_{s=1}^{r_\infty} (j - \tau_s^{(\infty)}) \nu_s^{(\infty)} + \frac{1}{2} \sum_{s=1}^{r_\infty} \left\{ m(n-1) + \mu_s^{(\infty)} - 1 + \frac{1}{\nu_s^{(\infty)}} \right\} \nu_s^{(\infty)}. \quad (21)$$

We can then write

$$\phi^{(i)}(z, v) = \delta_1^{(k)} \phi_1^{(i_k)}(z, v) + \dots + \delta_{\rho_k}^{(k)} \phi_{\rho_k}^{(i_k)}(z, v), \quad (22)$$

$$\phi^{(i_\infty+j)}(\xi, \eta) = \delta_1^{(\infty)} \phi_1^{(i_\infty+j)}(\xi, \eta) + \dots + \delta_{\rho_\infty}^{(\infty)} \phi_{\rho_\infty}^{(i_\infty+j)}(\xi, \eta), \quad (23)$$

where the δ 's are arbitrary constants excepting for the conditions imposed upon them in assuming the function $H(z, v)$ to be simultaneously representable in the forms (14) and (15). Equating these two forms we have

$$P(z, v) + \sum_k \sum_{s=1}^{\rho_k} \frac{\delta_s^{(k)} \phi_s^{(i_k)}(z, v)}{(z - a_k)^{i_k}} = - \sum_{s=1}^{\rho_\infty} \frac{\delta_s^{(\infty)} \phi_s^{(i_\infty+j)}(\xi, \eta)}{\xi^{i_\infty}} + \xi^j((\xi, \eta)). \quad (24)$$

Here the polynomial $P(z, v)$ identifies itself with the integral portion of the expression $\xi^{-i_\infty} \phi^{(i_\infty+j)}(\xi, \eta)$ transformed to terms of (z, v) and therefore depends on the constants $\delta_s^{(\infty)}$. Furthermore, the totality of the conditions to which the constants δ must be subjected is obtained on transforming

$$- \xi^{-i_\infty} \sum_{s=1}^{\rho_\infty} \delta_s^{(\infty)} \phi_s^{(i_\infty+j)}(\xi, \eta) \quad (25)$$

to terms of (z, v) and identifying the terms in the transformed expression which involve negative powers of z , with the corresponding terms in the expression

$$\sum_k \sum_{s=1}^{\rho_k} \frac{\delta_s^{(k)} \phi_s^{(i_k)}(z, v)}{(z - a_k)^{i_k}} \quad (26)$$

after the coefficients of the several powers of v have been developed in powers of $\frac{1}{z}$. The developments here in question will evidently involve only negative powers of z . We desire then to study the conditions imposed on the constants δ by the statement of the identity

$$\sum_k \sum_{s=1}^{\rho_k} \frac{\delta_s^{(k)} \Phi_s^{(t_k)}(z, v)}{(z - \alpha_k)^{t_k}} + \sum_{s=1}^{\rho_\infty} \frac{\delta_s^{(\infty)} \Phi_s^{(t_\infty + j)}(\xi, \eta)}{\xi^{t_\infty}} = \text{Polynomial in } (z, v) + \xi^j((\xi, \eta)). \quad (27)$$

We have to suppose that the total expression on the left-hand side of this identity has been transformed to terms of (z, v) , and that it is then represented as a polynomial in v with coefficients which are developed in powers of $\frac{1}{z}$, the number of negative powers of $\frac{1}{z}$ involved being of course finite. The conditions to which the constants δ are subjected are then obtained on equating to 0 the aggregate coefficient of an element $z^{-r} v^{n-t}$, where t can have any one of the values $1, \dots, n$ and where in $z^{-r} v^{n-t} = \xi^{r-m(n-t)} \eta^{n-t}$ the integer r is positive and subject also to the condition $r - m(n - t) < j$.

In Chapter IX of my book I have introduced a certain polynomial $R(z, v)$, such that its product by the function $F(z, v)$ has identically the form

$$F(z, v) \cdot R(z, v) = v^{2n-1} + S_{n-1} v^{n-1} + \dots + S_0. \quad (28)$$

Here the coefficient of v^{2n-1} in the product requires the coefficient of v^{n-1} in $R(z, v)$ to be 1, and the function $R(z, v)$ is then completely determined by the vanishing of the coefficients of $v^{2n-2}, \dots, v^n$ in the product. The coefficients $S_{n-1}, \dots, S_0$ in the product are of course completely determined when $R(z, v)$ is determined. At the same time also we evidently have 1 for the coefficient of η^{n-1} in the function

$$\overline{R}(\xi, \eta) = \xi^{m(n-1)} R(z, v), \quad (29)$$

and terms involving the powers $\eta^{2n-2}, \dots, \eta^n$ are lacking in the product $G(\xi, \eta) \cdot \overline{R}(\xi, \eta)$.

Consider all the polynomials $\phi(z, v)$, reduced in v , which possess a certain set of adjoint orders of coincidence $\mu'_1, \dots, \mu'_r$ corresponding to a value $z = a$. Supposing these functions to be arranged according to powers of $z - a$ and v and discarding terms divisible by $(z - a)^t$, we obtain a system of functions of the type $\phi^{(t)}(z, v)$, and of these a certain number ρ will be linearly independent

of one another. A complete system of linearly independent functions of this type we indicate by the notation

$$\phi_s^{(i)}(z, v) \quad (s = 1, 2, \dots, \rho). \quad (30)$$

Multiply these functions by the function $R(z, v)$ and write the products in the form

$$R(z, v) \cdot \phi_s^{(i)}(z, v) = \theta_s(z, v) + v^{n-1} \zeta_s^{(i)}(z, v) \quad (s = 1, 2, \dots, \rho), \quad (31)$$

where in the element $v^{n-1} \zeta_s^{(i)}(z, v)$ are included all those terms of the product which are divisible by v^{n-1} and which are not at the same time divisible by $(z-a)^i$. Any term then in $\theta_s(z, v)$ which is divisible by v^{n-1} must also be divisible by $(z-a)^i$. At the same time we also have

$$F(z, v) \cdot \zeta_s^{(i)}(z, v) = \Pi_s(z, v) + v^n \phi_s^{(i)}(z, v) \quad (s = 1, 2, \dots, \rho), \quad (32)$$

where $v^n \phi_s^{(i)}(z, v)$ is constituted out of all those terms in the product on the left which are divisible by v^n and which are not divisible by $(z-a)^i$. The truth of this is seen on multiplying the identities (31) through by $F(z, v)$ and equating on the two sides those terms which are divisible by v^{2n-1} but not by $(z-a)^i$, bearing the while in mind the formula (28).

In like manner we might consider all those polynomials $\phi(\xi, \eta)$, reduced in η , which possess a certain set of adjoint orders of coincidence relative to the equation $G(\xi, \eta) = 0$ for the value $\xi = 0$. On discarding in these polynomials terms divisible by ξ^i we obtain a system of functions of the type $\phi^{(i)}(\xi, \eta)$, and of these a certain number ρ_∞ will be linearly independent of one another. A complete system of linearly independent functions of this type we might indicate by the notation

$$\phi_s^{(i)}(\xi, \eta) \quad (s = 1, 2, \dots, \rho_\infty). \quad (33)$$

Multiplying these functions by the function $\bar{R}(\xi, \eta)$ we can write the products in the form

$$\bar{R}(\xi, \eta) \cdot \phi_s^{(i)}(\xi, \eta) = \theta_s(\xi, \eta) + \eta^{n-1} \zeta_s^{(i)}(\xi, \eta) \quad (s = 1, 2, \dots, \rho_\infty), \quad (34)$$

where in the element $\eta^{n-1} \zeta_s^{(i)}(\xi, \eta)$ are included all those terms of the product which are divisible by η^{n-1} and which are not at the same time divisible by ξ^i . Any term then in $\theta_s(\xi, \eta)$ which is divisible by η^{n-1} must also be divisible by ξ^i . At the same time we also have

$$G(\xi, \eta) \cdot \zeta_s^{(i)}(\xi, \eta) = \Pi_s(\xi, \eta) + \eta^n \phi_s^{(i)}(\xi, \eta) \quad (s = 1, 2, \dots, \rho_\infty), \quad (35)$$

where $\eta^n \phi_s^{(i)}(\xi, \eta)$ is constituted out of all those terms in the product on the left which are divisible by η^n and which are not divisible by ξ^i . Relatively to the

equation $G(\xi, \eta) = 0$, and for the value $\xi = 0$, formulae (34) and (35) are the analogues of the formulae (31) and (32), respectively, for the equation $F(z, v) = 0$ relatively to the value $z = a$. When we make use of the expression $\theta_s(\xi, \eta)$, for example, in the one set of formulae, it will be understood of course that we are not dealing with the same form as that which was designated by $\theta_s(z, v)$ in the other set of formulae.

Now the necessary and sufficient conditions in order that an integral rational function $\psi(z, v)$ may be complementary adjoint to the order i to a system of integral rational functions $\phi(z, v)$, whose orders of coincidence with the branches of the several cycles corresponding to the value $z = a$ are not less than the respective members of a set of adjoint numbers $\mu'_1, \dots, \mu'_r$, are obtained* on equating to 0 the coefficients of $(z - a)^{i-1}v^{n-1}$ in the ρ products

$$\zeta_s^{(i)}(z, v) \cdot \psi(z, v) \quad (s = 1, 2, \dots, \rho). \quad (36)$$

This is evidently equivalent to saying that the necessary and sufficient conditions in order that the orders of coincidence of the integral rational function $\psi(z, v)$ may be complementary adjoint to the orders of coincidence of the system of functions $(z - a)^{-i} \phi(z, v)$, are obtained on equating to 0 the coefficients of $(z - a)^{-1}v^{n-1}$ in the ρ products

$$(z - a)^{-i} \zeta_s^{(i)}(z, v) \cdot \psi(z, v) \quad (s = 1, 2, \dots, \rho). \quad (37)$$

The coefficients here in question are however the residues of the coefficients of v^{n-1} relative to the value $z = a$ in the ρ products (37). Now in the rational functions of z which constitute the coefficients of the several powers of v in these products the only poles which can present themselves are evidently $z = a$ and $z = \infty$, and the only residues which appear must correspond to these values. Since, however, the sum of the residues of a rational function of z is zero, if the residues of the coefficients of v^{n-1} in the products (37) relative to the value $z = a$ vanish, so also will the residues relative to the value $z = \infty$ vanish, and conversely.

We may then say that the necessary and sufficient conditions in order that the orders of coincidence of an integral rational function $\psi(z, v)$ for the value $z = a$ may be complementary adjoint to the orders of coincidence of the system of functions $(z - a)^{-i} \phi(z, v)$, in which the numerator $\phi(z, v)$ possesses a certain system of adjoint orders of coincidence, are obtained on equating to 0 the residues relative to the value $z = \infty$ of the coefficients of v^{n-1} in the ρ products (37).

* A. F., IX.

In like manner we see that the necessary and sufficient conditions in order that the orders of coincidence of an integral rational function $\bar{\Psi}(\xi, \eta)$, relative to the equation $G(\xi, \eta) = 0$ for the value $\xi = 0$, may be complementary adjoint to the orders of coincidence of the system of functions $\xi^{-(i_\infty+j)} \phi(\xi, \eta)$, in which the numerator $\phi(\xi, \eta)$ possesses the system of adjoint orders of coincidence $i_\infty + \tau_1^{(\infty)}, \dots, i_\infty + \tau_{r_\infty}^{(\infty)}$, are obtained on equating to 0 the residues relative to the value $\xi = 0$ (or $\xi = \infty$) of the coefficients of η^{n-1} in the ρ_∞ products

$$\xi^{-(i_\infty+j)} \zeta_s^{(i_\infty+j)}(\xi, \eta) \cdot \bar{\Psi}(\xi, \eta) \quad (s = 1, 2, \dots, \rho_\infty). \quad (38)$$

This is evidently equivalent to the statement that the necessary and sufficient conditions in order that the orders of coincidence of an integral rational function $\bar{\Psi}(\xi, \eta)$, relative to the equation $G(\xi, \eta) = 0$ for the value $\xi = 0$, may not be less than the numbers

$$j - \tau_1^{(\infty)} + m(n-1) + \mu_1^{(\infty)} - 1 + \frac{1}{v_1^{(\infty)}}, \dots, j - \tau_{r_\infty}^{(\infty)} + m(n-1) + \mu_{r_\infty}^{(\infty)} - 1 + \frac{1}{v_{r_\infty}^{(\infty)}}$$

respectively, are obtained on equating to 0 the coefficients of $\xi^{-1} \eta^{n-1}$ in the ρ_∞ products (38).

Again this statement is equivalent to saying that the necessary and sufficient conditions in order that the orders of coincidence, relative to the equation $G(\xi, \eta) = 0$, of a function

$$\xi^{-j-m(n-1)} \bar{\Psi}(\xi, \eta),$$

for the value $\xi = 0$, may not be less than the numbers

$$-\tau_1^{(\infty)} + \mu_1^{(\infty)} - 1 + \frac{1}{v_1^{(\infty)}}, \dots, -\tau_{r_\infty}^{(\infty)} + \mu_{r_\infty}^{(\infty)} - 1 + \frac{1}{v_{r_\infty}^{(\infty)}}$$

respectively, are obtained on equating to 0 the coefficients of $z^{-m(n-1)+1} v^{n-1}$ in the ρ_∞ products (38) transformed to terms of (z, v) . We conclude therefore that the necessary and sufficient conditions in order that the orders of coincidence of a function $\xi^{-j-m(n-1)} \bar{\Psi}(\xi, \eta)$ transformed to terms of (z, v) , for the value $z = \infty$, may be complementary adjoint relatively to the equation $F(z, v) = 0$, to the orders of coincidence $\tau_1^{(\infty)}, \dots, \tau_{r_\infty}^{(\infty)}$ of the system of functions $\xi^{-i_\infty} \phi(\xi, \eta)$, are obtained on equating to 0 the coefficients of $z^{-1} v^{n-1}$ in the ρ_∞ products

$$\xi^{-i_\infty} \zeta_s^{(i_\infty+j)}(\xi, \eta) \cdot \xi^{-j-m(n-1)+2} \bar{\Psi}(\xi, \eta) \quad (s = 1, 2, \dots, \rho_\infty),$$

transformed to terms of (z, v) .

By choosing j sufficiently large the expression $\xi^{-j-m(n-1)+2} \bar{\Psi}(\xi, \eta)$ will become a polynomial in (z, v) , and conversely any polynomial in (z, v) can be written in this form on choosing j sufficiently large.

Writing therefore

$$\xi^{-j-m(n-1)+2}\overline{\psi}(\xi, \eta) = \psi(z, v)$$

a polynomial in (z, v) , we derive that the necessary and sufficient conditions, in order that the orders of coincidence of the function $z^2\psi(z, v)$ for the value $z = \infty$ may be complementary adjoint, relatively to the equation $F(z, v) = 0$, to the orders of coincidence $\tau_1^{(\infty)}, \dots, \tau_{r_\infty}^{(\infty)}$, are obtained on equating to 0 the coefficients of $z^{-1}v^{n-1}$ in the ρ_∞ products

$$\xi^{-i_\infty} \zeta_s^{(i_\infty+j)}(\xi, \eta) \cdot \psi(z, v) \quad (s = 1, 2, \dots, \rho_\infty), \quad (39)$$

transformed to terms of (z, v) , the integer j being chosen sufficiently large.

We may then say that the necessary and sufficient conditions, in order that the orders of coincidence of the polynomial $\psi(z, v)$ for the value $z = \infty$ may be complementary adjoint to the order 2 to the orders of coincidence $\tau_1^{(\infty)}, \dots, \tau_{r_\infty}^{(\infty)}$ of a system of functions $\xi^{-i_\infty} \phi(\xi, \eta)$, are obtained on equating to 0 the residues relative to the value $z = \infty$ of the coefficient of v^{n-1} in the ρ_∞ products (39), the integer j being chosen large enough.

We shall now return to the identity (27) and to the study of the conditions which its existence imposes on the constants δ . Multiply the identity through by the function

$$R(z, v) = \xi^{-m(n-1)} \overline{R}(\xi, \eta).$$

Replace the products

$$R(z, v) \cdot \phi_s^{(i_k)}(z, v)$$

by expressions of the type given on the right-hand side of formula (31), and the products

$$\overline{R}(\xi, \eta) \cdot \phi_s^{(i_\infty+j)}(\xi, \eta)$$

by expressions of the type given on the right-hand side of formula (34), on replacing i by $i_\infty + j$. In the resulting identity retain only those terms which are divisible by v^{n-1} or η^{n-1} , and we evidently obtain an identity of the form

$$\begin{aligned} v^{n-1} \sum_{k=1}^{\rho_k} \frac{\delta_s^{(k)} \zeta_s^{(i_k)}(z, v)}{(z - a_k)^{i_k}} + \xi^{-m(n-1)} \eta^{n-1} \sum_{s=1}^{\rho_\infty} \frac{\delta_s^{(\infty)} \zeta_s^{(i_\infty+j)}(\xi, \eta)}{\xi^{i_\infty}} \\ = \text{Product of } v^{n-1} \text{ and a polynomial in } (z, v) + \xi^{-m(n-1)} \eta^{n-1} \xi^j ((\xi, \eta))_1. \end{aligned}$$

Dividing through by $v^{n-1} = \xi^{-m(n-1)} \eta^{n-1}$ this identity becomes

$$\begin{aligned} \sum_{k=1}^{\rho_k} \frac{\delta_s^{(k)} \zeta_s^{(i_k)}(z, v)}{(z - a_k)^{i_k}} + \sum_{s=1}^{\rho_\infty} \frac{\delta_s^{(\infty)} \zeta_s^{(i_\infty+j)}(\xi, \eta)}{\xi^{i_\infty}} \\ = \text{Polynomial in } (z, v) + \xi^j ((\xi, \eta))_1. \end{aligned} \quad (40)$$

Now multiply this identity through by the function

$$F(z, v) = \xi^{-mn} G(\xi, \eta).$$

Replace the products

$$F(z, v) \cdot \zeta_s^{(i_k)}(z, v)$$

by expressions of the type given on the right-hand side of formula (32), and the products

$$G(\xi, \eta) \cdot \zeta_s^{(i_\infty+j)}(\xi, \eta)$$

by expressions of the type given on the right-hand side of formula (35). In the resulting identity retain only those terms which are divisible by v^n or η^n , and we arrive at an identity of the form

$$\begin{aligned} v^n \sum_k \sum_{s=1}^{\rho_k} \frac{\delta_s^{(k)} \phi_s^{(i_k)}(z, v)}{(z - a_k)^{i_k}} + \xi^{-mn} \eta^n \sum_{s=1}^{\rho_\infty} \frac{\delta_s^{(\infty)} \phi_s^{(i_\infty+j)}(\xi, \eta)}{\xi^{i_\infty}} \\ = \text{Product of } v^n \text{ and a polynomial in } (z, v) + \xi^{-mn+j} \eta^n ((\xi, \eta))_2. \end{aligned}$$

Dividing through by $v^n = \xi^{-mn} \eta^n$ this identity becomes

$$\sum_k \sum_{s=1}^{\rho_k} \frac{\delta_s^{(k)} \phi_s^{(i_k)}(z, v)}{(z - a_k)^{i_k}} + \sum_{s=1}^{\rho_\infty} \frac{\delta_s^{(\infty)} \phi_s^{(i_\infty+j)}(\xi, \eta)}{\xi^{i_\infty}} = \text{Polynomial in } (z, v) + \xi^j ((\xi, \eta))_2.$$

This last identity; at least so far as the constants δ are concerned, is equivalent to the identity (27), and it follows from the identity (40). The identity (40) is however a consequence of the identity (27), and therefore imposes no other conditions on the constants than those which are involved in the identity (27). The conditions imposed on the constants δ by the identities (27) and (40) are therefore precisely the same, and for the study of these conditions we can consequently make use of the identity (40) instead of employing the identity (27).

On the left-hand side of the identity (40) equate to 0 the coefficients of terms in

$$z^{-r} v^{n-t} = \xi^{r-(n-t)m} \eta^{n-t} \quad (0 < r < j + (n-t)m; t = 1, 2, \dots, n). \quad (41)$$

This gives us a number of linear equations which must be satisfied by the constants δ . Of these a certain number will be independent, and the remaining ones will follow as a consequence of these. Suppose we represent by $\alpha_{r-1, t-1}$ the multipliers of the respective equations which give the linear connection between a dependent equation and the independent equations. Construct the polynomial

$$\psi(z, v) = \sum_{r, t} \alpha_{r-1, t-1} z^{r-1} v^{t-1}, \quad (42)$$

and the dependence of the one equation on the others is evidently equivalent to the statement that the coefficient of $z^{-1} v^{n-1}$ is equal to 0 in each one of the products

$$\begin{aligned} (z - a_k)^{-i_k} \zeta_s^{(i_k)}(z, v) \cdot \psi(z, v) & \quad (s = 1, 2, \dots, \rho_k), \\ \xi^{-i_\infty} \zeta_s^{(i_\infty+j)}(\xi, \eta) \cdot \psi(z, v) & \quad (s = 1, 2, \dots, \rho_\infty). \end{aligned}$$

These conditions, however, from what we have seen in the preceding, are those which are necessary and sufficient in order that the polynomial $\psi(z, v)$ may be complementary adjoint to the orders of coincidence $-\sigma_1^{(k)}, -\sigma_2^{(k)}, \dots, -\sigma_{r_k}^{(k)}$ corresponding to the finite values of the variable $z = a_k$, and in order that it may be complementary adjoint to the order 2 to the orders of coincidence $\tau_1^{(\infty)}, \tau_2^{(\infty)}, \dots, \tau_{r_\infty}^{(\infty)}$ corresponding to the value $z = \infty$. Furthermore, among the conditions to which we subject the constants δ , the number which are not independent of one another is evidently equal to the number of arbitrary coefficients involved in the most general polynomial

$$\psi(z, v) = \sum_{r, t} \alpha_{r-1, t-1} z^{r-1} v^{t-1} \quad (0 < r < j + m(n-t); t = 1, 2, \dots, n), \quad (43)$$

whose orders of coincidence for the finite values $z = a_k$ do not fall short of the numbers

$$\sigma_1^{(k)} + \mu_1^{(k)} - 1 + \frac{1}{v_1^{(k)}}, \dots, \sigma_{r_k} + \mu_{r_k} - 1 + \frac{1}{v_{r_k}^{(k)}} \quad (44)$$

respectively, while its orders of coincidence for the value $z = \infty$ are not less than the numbers

$$-\tau_1^{(\infty)} + \mu_1^{(\infty)} + 1 + \frac{1}{v_1^{(\infty)}}, \dots, -\tau_{r_\infty}^{(\infty)} + \mu_{r_\infty}^{(\infty)} + 1 + \frac{1}{v_{r_\infty}^{(\infty)}} \quad (45)$$

respectively. The total number of arbitrary constants involved in the expression of the most general polynomial $\psi(z, v)$ of the character here in question we designate by the symbol N_ψ . The total number of equations to which we subject the constants δ is given by the summation

$$(j-1)n + \sum_{t=1}^n m(n-t) = (j-1)n + \frac{1}{2}mn(n-1),$$

and the number of linearly independent equations satisfied by the constants δ is therefore

$$(j-1)n + \frac{1}{2}mn(n-1) - N_\psi. \quad (46)$$

The total number of the constants δ , as we see on referring to formulae (19) and (21), is

$$\sum_k \rho_k = \frac{1}{2}mn(n-1) + jn + \frac{1}{2} \sum_{k=1}^{r_k} \left(\mu_s^{(k)} - 1 + \frac{1}{v_s^{(k)}} \right) v_s^{(k)} + \sum'_{k=1} \sum_{s=1}^{r_k} \sigma_s^{(k)} v_s^{(k)} - \sum_{s=1}^{r_\infty} \tau_s^{(\infty)} v_s^{(\infty)}, \quad (47)$$

where the accented summation with regard to k is supposed to extend only to finite values $z = a_k$, whereas the summation without the accent extends also to the value $z = \infty$.

We have now come back to the reasoning employed about the middle of Chapter XII of "The Theory of the Algebraic Functions," and from here on

we shall follow the text in developing the subject. The number of the constants δ which remain arbitrary on solving the system of equations to which we have subjected them, is obtained on subtracting the number given in formula (46) from that given in formula (47). This gives us the expression

$$N_\psi + n + \frac{1}{2} \sum_{k=1}^{r_k} \sum_{s=1} \left(\mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}} \right) \nu_s^{(k)} + \sum' \sum_{k=1}^{r_k} \sigma_s^{(k)} \nu_s^{(k)} - \sum_{s=1}^{r_\infty} \tau_s^{(\infty)} \nu_s^{(\infty)} \quad (48)$$

for the number of the arbitrary constants δ , and therefore for the number of the independent* arbitrary constants involved in the coefficients of the most general rational function $H(z, v)$ which becomes infinite to orders not exceeding $\sigma_1^{(k)}, \sigma_2^{(k)}, \dots, \sigma_{r_k}^{(k)}$ respectively, for the branches of the several cycles corresponding to the various finite values of the variable $z = a_k$, while its orders of coincidence with the branches of the several cycles corresponding to the value $z = \infty$ do not fall short of the numbers $\tau_1^{(\infty)}, \tau_2^{(\infty)}, \dots, \tau_{r_\infty}^{(\infty)}$ respectively. On employing the symbol N_H to designate the number of arbitrary constants involved in the expression of the most general function $H(z, v)$ of the character here in question, and on replacing the symbols $-\sigma_1^{(k)}, -\sigma_2^{(k)}, \dots, \sigma_{r_k}^{(k)}$ in the expression (48) by the symbols $\tau_1^{(k)}, \tau_2^{(k)}, \dots, \tau_{r_k}^{(k)}$, we have

$$N_H = N_\psi + n + \frac{1}{2} \sum_{k=1}^{r_k} \sum_{s=1} \left(\mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}} \right) \nu_s^{(k)} - \sum_{k=1}^{r_k} \sum_{s=1} \tau_s^{(k)} \nu_s^{(k)}. \quad (49)$$

Now a rational function of (z, v) which is adjoint for all finite values of the variable z must be an integral function of these variables.† Therefore a rational function of (z, v) possessing for finite values of the variable z the orders of coincidence given in (44) must be a polynomial in (z, v) . Also a polynomial in (z, v) possessing for the value $z = \infty$ the orders of coincidence given in (45), will be included in the form given in (43) when j has been chosen large enough.‡ The polynomial $\psi(z, v)$ here in question, then, may be assumed to be the most general rational function whose orders of coincidence for finite values of the variable $z = a_k$ do not fall short of the numbers given in (44), while its orders of coincidence for the value $z = \infty$ are not less than the numbers given in (45).

On employing for brevity the notation $\bar{\tau}_1^{(k)}, \bar{\tau}_2^{(k)}, \dots, \bar{\tau}_{r_k}^{(k)}$ to designate the numbers in (44) and the notation $\bar{\tau}_1^{(\infty)}, \bar{\tau}_2^{(\infty)}, \dots, \bar{\tau}_{r_\infty}^{(\infty)}$ to designate the numbers in (45), we see that $\psi(z, v)$ is the most general rational function of (z, v) built on the basis $(\bar{\tau})$, while $H(z, v)$ is the most general rational function built on the

* See *A. F.*, pp. 128, 129.

† See *A. F.*, V, p. 34.

‡ See *A. F.*, pp. 126, 127.

basis (τ) . Here the bases (τ) and $(\bar{\tau})$ are complementary to each other with regard to the level furnished by the function $F'_v(z, v)$, for we have

$$\tau_s^{(k)} + \bar{\tau}_s^{(k)} = \mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}} \quad (s = 1, 2, \dots, r_k) \quad (50)$$

for all finite values of the variable $z = a_k$, and for the value $z = \infty$ we have

$$\tau_s^{(\infty)} + \bar{\tau}_s^{(\infty)} = \mu_s^{(\infty)} + 1 + \frac{1}{\nu_s^{(\infty)}} \quad (s = 1, 2, \dots, r_\infty). \quad (51)$$

The only limitation which we have here placed on the basis (τ) is that none of its numbers corresponding to any finite value of the variable z shall be positive. This limitation, however, is readily removed, as has been done in the text.* Assume, namely, that (τ) is any arbitrary basis of coincidences and $(\bar{\tau})$ the complementary basis whose coincidences satisfy the relations (50) and (51), and designate by $H(z, v)$ and $\bar{H}(z, v)$ respectively the most general rational functions built on these bases. Where $P(z)$ is any specific polynomial in z , the functions $\frac{H(z, v)}{P(z)}$ and $P(z) \bar{H}(z, v)$ are evidently also the most general rational functions built on certain complementary bases. Also by proper choice of $P(z)$ the basis for the former function will include no positive numbers corresponding to finite values of the variable z , and the complementary bases here in question will then come under the case for which we have already proved the complementary theorem. Furthermore the numbers of arbitrary constants involved in the expression of the functions $H(z, v)$ and $\bar{H}(z, v)$ are evidently the same as the numbers of arbitrary constants involved in the expressions of the functions $\frac{H(z, v)}{P(z)}$ and $P(z) \bar{H}(z, v) = \psi(z, v)$, while the sum of the orders of coincidence of the function $P(z)$ for all values of the variable z and for all the cycles corresponding is 0, and we therefore immediately derive

$$N_H = N_{\bar{H}} + n + \frac{1}{2} \sum_k \sum_{s=1}^{r_k} \left(\mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}} \right) \nu_s^{(k)} - \sum_k \sum_{s=1}^{r_k} \tau_s^{(k)} \nu_s^{(k)}, \quad (52)$$

where (τ) and $(\bar{\tau})$ are any complementary bases whose coincidences satisfy the relations (50) and (51). By combination of (52) with the relations (50) and (51) we can readily write our result in the form

$$N_H + \frac{1}{2} \sum_k \sum_{s=1}^{r_k} \tau_s^{(k)} \nu_s^{(k)} = N_{\bar{H}} + \frac{1}{2} \sum_k \sum_{s=1}^{r_k} \bar{\tau}_s^{(k)} \nu_s^{(k)},$$

and the proof is immediate that this formula holds also when the relations (50) and (51) are replaced by the more general relations given in (2) and (3).

* *A. F.*, pp. 134, 135.